3+1 Splitting of the Generalized Harmonic Equations

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EGM
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Numerical Relativity

Interpret general relativity as an initial value problem: Split spacetime derivatives into time and space derivatives

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$$\swarrow \quad \searrow$$

$$\partial_t \qquad\qquad \partial_i$$

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- $\dot{g}$–$\dot{K}$
- ADM
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$$g_{\mu\nu} \quad \swarrow \downarrow \searrow \quad \alpha \qquad \beta^i \qquad g_{ij}$$

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Generalized Harmonic formulation can be interpreted as an initial value problem without splitting tensor indices.

Einstein Equations

$$R_{\mu\nu} = 0$$

Generalized Harmonic Eqs.

$$\begin{aligned} R_{\mu\nu} &= \nabla_{(\mu}\mathcal{C}_{\nu)} \\ \mathcal{C}^{\mu} &\equiv H^{\mu} + \Gamma^{\mu}_{\sigma\rho} g^{\sigma\rho} = 0 \end{aligned}$$

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Initial Value Problem

$$\begin{aligned} \partial_t g_{ij} &= \mathcal{L}_\beta g_{ij} - 2\alpha K \\ \partial_t K_{ij} &= \mathcal{L}_\beta K_{ij} + \alpha K K_{ij} + \cdots \\ \mathcal{H} &\equiv K^2 - K^{ij} K_{ij} + R = 0 \\ \mathcal{M}_i &\equiv D_j K^j_i - D_i K = 0 \end{aligned}$$

- $\mathcal{H} = \mathcal{M}_i = 0$ preserved in time if $\mathcal{H} = \mathcal{M}_i = 0$ initially
- Generate solution by evolving g_{ij} and K_{ij} with α and β^i chosen freely

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$$\begin{aligned} R_{\mu\nu} &= \nabla_{(\mu} \mathcal{C}_{\nu)} \\ \mathcal{C}^\mu &\equiv H^\mu + \Gamma^\mu_{\sigma\rho} g^{\sigma\rho} = 0 \end{aligned}$$

Initial Value Problem

$$\begin{aligned} \Box g_{\mu\nu} &= -2\partial_{(\mu} H_{\nu)} + \cdots \\ \Box \mathcal{C}_\mu &= -\mathcal{C}^\nu \nabla_{(\mu} \mathcal{C}_{\nu)} \end{aligned}$$

- $\mathcal{C}_\mu = 0$ preserved in time if $\mathcal{C}_\mu = 0$ and $\partial_t \mathcal{C}_\mu = 0$ (equiv. to $\mathcal{H} = \mathcal{M}_i = 0$) initially
- Generate solution by evolving $g_{\mu\nu}$ with H_μ chosen freely

Goal:

Write generalized harmonic (GH) equations with 3+1 splitting of derivatives *and* tensor indices:

$$
\begin{aligned}
\partial_t g_{ij} &= \ldots \\
\partial_t K_{ij} &= \ldots \\
\partial_t \alpha &= \ldots \\
\partial_t \beta^i &= \ldots \\
&\vdots
\end{aligned}
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Motivation:

- GH is expressed in the same language/notation as $\dot{g}$–$\dot{K}$, ADM, BSSN, *etc.* Comparison and insights.

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Motivation:

- GH is expressed in the same language/notation as $\dot{g}$–$\dot{K}$, ADM, BSSN, *etc.* Comparison and insights.
- Result is *nice*.

3+1 Splitting of GH: technical details

Covariant GH equations:

$$\begin{aligned} R_{\mu\nu} &= \nabla_{(\mu}\mathcal{C}_{\nu)} \\ \mathcal{C}^{\mu} &\equiv H^{\mu} + \Delta\Gamma^{\mu}_{\sigma\rho} g^{\sigma\rho} = 0 \end{aligned}$$

where H^{μ} is a vector and

$$\Delta\Gamma^{\mu}_{\sigma\rho} \equiv \Gamma^{\mu}_{\sigma\rho} - \tilde{\Gamma}^{\mu}_{\sigma\rho} = \frac{1}{2} g^{\mu\nu} \left(\tilde{\nabla}_{\sigma} g_{\rho\nu} + \tilde{\nabla}_{\rho} g_{\sigma\nu} - \tilde{\nabla}_{\nu} g_{\sigma\rho} \right)$$

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- Split $R_{\mu\nu}$, $\Gamma^{\mu}_{\sigma\rho}$, $\tilde{\Gamma}^{\mu}_{\sigma\rho}$, H_{μ}, $\nabla_{\mu}\mathcal{C}_{\nu}$
- Absorb terms $\sim F(g_{\mu\nu}, \tilde{g}_{\mu\nu}, \partial_{\sigma}\tilde{g}_{\mu\nu})$ into $H_{\perp}$ and H_i
- Define

$$\begin{aligned} K_{ij} &\equiv -(\partial_t g_{ij} - \mathcal{L}_{\beta} g_{ij})/(2\alpha) \\ \pi &\equiv (\partial_t \alpha - \beta^i \tilde{D}_i \alpha)/\alpha^2 + H_{\perp} \\ \rho^i &\equiv (\partial_t \beta^i - \beta^k \tilde{D}_k \beta^i)/\alpha^2 + D^i\alpha/\alpha - H^i \end{aligned}$$

GH Equations in 3+1 form ($\partial_\perp \equiv \partial_t - \mathcal{L}_\beta$)

$$
\begin{aligned}
\partial_\perp g_{ij} &= -2\alpha K_{ij} \\
\partial_\perp K_{ij} &= \alpha \left[R_{ij} - 2K_{ik}K^k_j + KK_{ij} \right] - D_i D_j \alpha - \alpha \mathcal{C}_\perp K_{ij} - \alpha D_{(i}\mathcal{C}_{j)} \\
\partial_\perp \alpha &= \alpha^2 \pi - \alpha^2 H_\perp \\
\partial_t \beta^i &= \beta^j \tilde{D}_j \beta^i + \alpha^2 \rho^i - \alpha D^i \alpha + \alpha^2 H^i \\
\partial_\perp \pi &= -\alpha K_{ij} K^{ij} + D_i D^i \alpha + \mathcal{C}^i D_i \alpha \\
\partial_\perp \rho^i &= \alpha D^i \pi - \pi D^i \alpha - 2K^{ij} D_j \alpha + 2\alpha K^{jk} \Delta\Gamma^i{}_{jk} + g^{k\ell} \tilde{D}_k \tilde{D}_\ell \beta^i
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C}_\perp &\equiv \pi + K \\
\mathcal{C}^i &\equiv -\rho^i + \Delta\Gamma^i_{jk} g^{jk} \\
\mathcal{H} &\equiv K^2 - K_{ij}K^{ij} + R \\
\mathcal{M}_i &\equiv D_j K^j_i - D_i K
\end{aligned}
$$

Constraint Evolution

$$
\begin{aligned}
\partial_\perp \mathcal{C}_\perp &= -\alpha K \mathcal{C}_\perp + \alpha \mathcal{H} + \mathcal{C}^i D_i \alpha - \alpha D_i \mathcal{C}^i \\
\partial_\perp \mathcal{C}_i &= \mathcal{C}_\perp D_i \alpha - \alpha D_i \mathcal{C}_\perp - 2\alpha \mathcal{M}_i - 2\alpha K_{ij} \mathcal{C}^j \\
\partial_\perp \mathcal{H} &= -2\alpha\pi \mathcal{H} + 2\alpha R \mathcal{C}_\perp + 2\alpha (K^{ij} - K g^{ij}) D_i \mathcal{C}_j \\
&\quad -4 \mathcal{M}_i D^i \alpha - 2\alpha D^i \mathcal{M}_i \\
\partial_\perp \mathcal{M}_i &= -\mathcal{H} D_i \alpha + (K \delta_i^j - K_i^j) D_j (\alpha \mathcal{C}_\perp) - \frac{1}{2} \alpha D_i \mathcal{H} \\
&\quad -\alpha\pi \mathcal{M}_i + D^j \alpha D_{[i} \mathcal{C}_{j]} + D_i (\alpha D_j \mathcal{C}^j) \\
&\quad -\frac{1}{2} \alpha R_{ij} \mathcal{C}^j - \alpha D^j D_j \mathcal{C}_i
\end{aligned}
$$

Now What?

- Direct comparison of GH with $\dot{g}$–$\dot{K}$, ADM, BSSN, *etc*
- Which terms are responsible for stable evolution?
- Moving puncture coordinates/evolution in GH?
- New formulations?

Symmetric Hyperbolicity

Conserved energy:

$$
\begin{aligned}
\varepsilon \;=\;& M^{ijk\ell}\left[\frac{1}{4}g^{mn}\partial_m g_{ij}\,\partial_n g_{k\ell} + (K_{ij} - \partial_i\beta_j/\alpha)(K_{k\ell} - \partial_k\beta_\ell/\alpha)\right] \\
& + N^{ij}\left[\frac{1}{\alpha^2}g^{k\ell}\partial_k\beta_i\,\partial_\ell\beta_j + (\rho_i - \partial_i\alpha/\alpha)(\rho_j - \partial_j\alpha/\alpha)\right] \\
& + C\left[\pi\pi + \frac{1}{\alpha^2}g^{ij}\partial_i\alpha\,\partial_j\alpha\right]
\end{aligned}
$$

where $M^{ijk\ell}$, N^{ij} and C are positive definite.

GH with Constraint Damping

$$R_{\mu\nu} - \nabla_{(\mu}\mathcal{C}_{\nu)} + \kappa\left[n_{(\mu}\mathcal{C}_{\nu)} - g_{\mu\nu}n^{\sigma}\mathcal{C}_{\sigma}/2\right] = 0$$

$\Longrightarrow$

$$\begin{aligned}
\partial_\perp g_{ij} &= -2\alpha K_{ij} \\
\partial_\perp K_{ij} &= \alpha\left[R_{ij} - 2K_{ik}K^k_j + KK_{ij}\right] - D_i D_j \alpha \\
&\quad -\alpha\mathcal{C}_\perp K_{ij} - \alpha D_{(i}\mathcal{C}_{j)} - \frac{1}{2}\kappa g_{ij}\mathcal{C}_\perp \\
\partial_\perp \alpha &= \alpha^2\pi - \alpha^2 H_\perp \\
\partial_t \beta^i &= \beta^j \tilde{D}_j \beta^i + \alpha^2\rho^i - \alpha D^i\alpha + \alpha^2 H^i \\
\partial_\perp \pi &= -\alpha K_{ij}K^{ij} + D_i D^i \alpha + \mathcal{C}^i D_i \alpha - \frac{1}{2}\kappa\alpha\mathcal{C}_\perp \\
\partial_\perp \rho^i &= \alpha D^i \pi - \pi D^i \alpha - 2K^{ij}D_j\alpha + 2\alpha K^{jk}\Delta\Gamma^i{}_{jk} \\
&\quad + g^{k\ell}\tilde{D}_k\tilde{D}_\ell\beta^i + \kappa\alpha\mathcal{C}^i
\end{aligned}$$

GH with Constraint Damping

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\begin{aligned}
\partial_\perp \mathcal{C}_\perp &= -\alpha K \mathcal{C}_\perp + \alpha \mathcal{H} + \mathcal{C}^i D_i \alpha - \alpha D_i \mathcal{C}^i - 2\kappa\alpha\mathcal{C}_\perp \\
\partial_\perp \mathcal{C}_i &= \mathcal{C}_\perp D_i \alpha - \alpha D_i \mathcal{C}_\perp - 2\alpha \mathcal{M}_i - 2\alpha K_{ij}\mathcal{C}^j - \kappa\alpha\mathcal{C}_i \\
\partial_\perp \mathcal{H} &= -2\alpha\pi\mathcal{H} + 2\alpha R \mathcal{C}_\perp + 2\alpha(K^{ij} - K g^{ij}) D_i \mathcal{C}_j \\
&\quad -4\mathcal{M}_i D^i \alpha - 2\alpha D^i \mathcal{M}_i - 2\kappa\alpha\mathcal{C}_\perp \\
\partial_\perp \mathcal{M}_i &= -\mathcal{H} D_i \alpha + (K\delta_i^j - K_i^j) D_j(\alpha\mathcal{C}_\perp) - \frac{1}{2}\alpha D_i \mathcal{H} \\
&\quad -\alpha\pi\mathcal{M}_i + D^j \alpha D_{[i}\mathcal{C}_{j]} + D_i(\alpha D_j \mathcal{C}^j) \\
&\quad -\frac{1}{2}\alpha R_{ij}\mathcal{C}^j - \alpha D^j D_j \mathcal{C}_i + \kappa D_i(\alpha\mathcal{C}_\perp)
\end{aligned}
$$