Hidden momentum, field momentum, and electromagnetic impulse

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(Received 1 February 2009; accepted 20 May 2009)

Electromagnetic fields carry energy, momentum, and angular momentum. The momentum density, $\epsilon_0(\mathbf{E}\times\mathbf{B})$, accounts (among other things) for the pressure of light. But even *static* fields can carry momentum, and this would appear to contradict a general theorem that the total momentum of a closed system is zero if its center of energy is at rest. In such cases, there must be some other (nonelectromagnetic) momenta that cancel the field momentum. What is the nature of this "hidden momentum" and what happens to it when the electromagnetic fields are turned off? © *2009 American Association of Physics Teachers.*

[DOI: 10.1119/1.3152712]

I. INTRODUCTION

The linear momentum density carried by electromagnetic fields is related to the Poynting vector[1]

$$\wp_{\rm em} = \frac{1}{c^2}\mathbf{S} = \epsilon_0(\mathbf{E}\times\mathbf{B}). \tag{1}$$

The classic example is an electromagnetic wave (see Fig. 1). When the wave strikes an absorber, its momentum is passed along in the form of the pressure of light. But there are other examples in which the fields are perfectly *static*, and yet the electromagnetic momentum is not zero. Consider, for instance, the following configurations.

Capacitor in a magnetic field. A charged parallel-plate capacitor (with uniform electric field $\mathbf{E}=-E\hat{\mathbf{y}}$) is placed in a uniform magnetic field $\mathbf{B}=B\hat{\mathbf{z}}$, as shown in Fig. 2.[2,3] Naively, the electromagnetic momentum is[4]

$$\mathbf{p}_{\rm em} = -\epsilon_0 EBAd\hat{\mathbf{x}} = -BQd\hat{\mathbf{x}}, \tag{2}$$

where A is the area of the plates, d is their separation, and Q is the charge on the upper plate.

Magnetic dipole and electric charge. A magnetic dipole $\mathbf{m}=m\hat{\mathbf{y}}$ is situated a distance a from a point charge q, as shown in Fig. 3.[5] The electromagnetic momentum is

$$\mathbf{p}_{\rm em} = \frac{\mu_0}{4\pi}\frac{qm}{a^2}\hat{\mathbf{x}} = \frac{1}{c^2}(\mathbf{E}\times\mathbf{m}), \tag{3}$$

where $\mathbf{E}$ is the electric field at the location of the dipole.

Polarized magnetized sphere. A sphere of radius R carries a uniform polarization $\mathbf{P}$ and a uniform magnetization $\mathbf{M}$ (see Fig. 4).[6] The momentum carried by the fields is

$$\mathbf{p}_{\rm em} = \tfrac{4}{9}\pi\mu_0 R^3(\mathbf{M}\times\mathbf{P}). \tag{4}$$

Coaxial cable. A long coaxial cable (length l) is connected to a battery of voltage V at one end and a resistor R at the other (see Fig. 5). The momentum carried by the fields is

$$\mathbf{p}_{\rm em} = \frac{lV^2}{c^2R}\hat{\mathbf{x}}. \tag{5}$$

It seems strange (to say the least!) for purely static fields to carry momentum. Can this possibly be the whole story? And what happens to the momentum when we turn off the fields? In Sec. II we explore the latter question in what would appear to be the simplest context: the parallel-plate capacitor in a uniform magnetic field. We are led to a surprising paradox. In Sec. III we return to the first question ("Is this the whole story?"), to which the answer is no. Here we develop the theory of "hidden momentum." In Sec. IV we work out the details for an electric dipole at the center of a spinning, uniformly charged spherical shell, and resolve the apparent paradox from Sec. II. In Sec. V we do the same for an electric dipole inside a long solenoid. In Sec. VI we demonstrate that hidden momentum always cancels electromagnetic momentum, in the static case,[7] and draw some general conclusions about the nature of the hidden momentum.

II. CAPACITOR IN A UNIFORM MAGNETIC FIELD

If the electric or magnetic field is turned off, the momentum originally stored in the fields must (one would think) be converted into ordinary mechanical momentum. For example, in the case of the capacitor in a magnetic field, we might connect a wire between the plates, allowing the capacitor to discharge slowly[8] (see Fig. 6). According to the Lorentz law, this wire will experience a force $F=IBd$ to the left, where I is the current in the wire. The net impulse delivered to the capacitor—which is to say, the mechanical momentum it acquires—is

$$\mathcal{I} = \int \mathbf{F}dt = -Bd\int\left(-\frac{dq}{dt}\right)dt\hat{\mathbf{x}} = Bd\int_Q^0 dq\hat{\mathbf{x}} = -BQd\hat{\mathbf{x}}, \tag{6}$$

which is precisely the momentum originally stored in the fields [see Eq. (2)].

Alternatively, we might turn off the magnetic field. According to Faraday's law, the changing magnetic field will induce an electric field,

$$\oint \mathbf{E}\cdot d\ell = -\frac{d\Phi}{dt} \tag{7}$$

(which, by Lenz's law, runs counterclockwise in Fig. 7). The x component of the force on the strip shown is $dF_x = E_x\sigma w dx$, where σ is the surface charge density, and the *net* force on the capacitor is $F_x=\sigma w\oint\mathbf{E}\cdot d\ell$, where the integral is taken clockwise around the dotted loop, and we have ignored the two short ends. The magnetic flux through this loop

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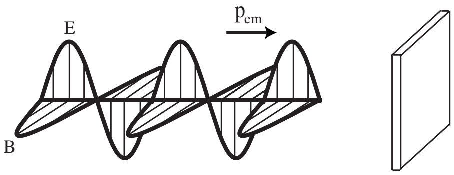

Fig. 1. An electromagnetic wave carries momentum in the direction of propagation; when it hits an absorber, this momentum is transferred in the form of the pressure of light.

(counting inward as positive, for consistency) is $\Phi = -Bld$, so $F_x = \sigma wld(dB/dt)$, and the impulse delivered to the capacitor is

$$\mathcal{I} = \int \mathbf{F} dt = \sigma wld \int \left(\frac{dB}{dt}\right) dt\hat{\mathbf{x}}$$

$$= \sigma wld \int_B^0 dB\hat{\mathbf{x}} = -BQd\hat{\mathbf{x}}, \tag{8}$$

the same as Eq. (6). Everything seems to be in order: when either the electric field is turned off (by discharging the capacitor) or the magnetic field is removed, the momentum originally stored in the fields is converted into ordinary mechanical momentum, and the capacitor moves off to the left (see Table I). Sounds good, but it is almost entirely wrong.[9]

In the first place, $\oint \mathbf{E}\cdot d\ell$ includes the two vertical segments, which do not contribute the force. Of course, we assume that the plates are very close together; doesn't that mean the "extra" piece is negligible? Unfortunately, it does not. Suppose we use a solenoid to establish the magnetic field (see Fig. 8). Because of the azimuthal symmetry, we can calculate the induced electric field explicitly, $E(2\pi r) = -\pi r^2 dB/dt$, which implies that

$$\mathbf{E} = -\frac{r}{2}\frac{dB}{dt}\hat{\phi} \tag{9}$$

and

$$\mathbf{E}\cdot d\ell = -\frac{r}{2}\frac{dB}{dt}(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}})\cdot(dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}) \tag{10a}$$

$$= \frac{1}{2}\frac{dB}{dt}(r\sin\phi dx - r\cos\phi dy)$$

$$= \frac{1}{2}\frac{dB}{dt}(ydx - xdy). \tag{10b}$$

So the force on the capacitor is

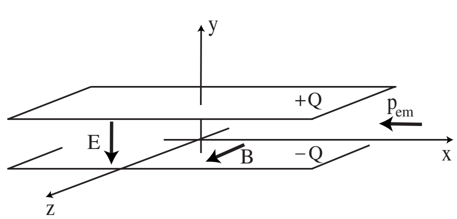

Fig. 2. A charged capacitor in the presence of an external magnetic field carries electromagnetic momentum even though nothing is moving.

$$F_x = \sigma w\left(\int_{\text{top}} \mathbf{E}\cdot d\ell - \int_{\text{bottom}} \mathbf{E}\cdot d\ell\right) \tag{11a}$$

$$= \sigma w\left[\frac{1}{2}\frac{dB}{dt}\frac{d}{2}l - \frac{1}{2}\frac{dB}{dt}\left(-\frac{d}{2}\right)l\right] = \frac{1}{2}\sigma wld\frac{dB}{dt}, \tag{11b}$$

which is *half* of what we obtained before. The two ends are shorter ($\int dy = d$), but they are farther out ($x = l/2$), and their contribution to $\oint \mathbf{E}\cdot d\ell$ is the *same* as the top and bottom; but they do not contribute to the force because there is no charge there. Apparently, the impulse delivered to the capacitor when $\mathbf{B}$ is turned off is *not* the same as the momentum originally stored in the fields.

But that is not all. Our naive expression for the momentum in the fields in Eq. (2) ignored the fringing field of the capacitor, and when this field is correctly included,[3] the answer is half as great (see Table II). Now lines 1 and 3 are consistent, but 2 is off! There is evidently a problem here, but it runs much deeper than that factor of 1/2, as we shall see in Sec. III.

III. HIDDEN MOMENTUM

There is a very general principle in special relativity,[10] which we shall call the *center of energy theorem*: if the center of energy[12] of a closed[13] system is at rest, then its total momentum is zero. The center of energy of a capacitor in a static magnetic field is certainly at rest, so if there is momentum in the *fields*, there must be some compensating *non*electromagnetic momentum elsewhere. Where is this "hidden momentum" located, and what is its nature? When we turn off $\mathbf{E}$ or $\mathbf{B}$, and the capacitor gets a kick, something else (in this case the solenoid) must get an equal and opposite kick so that the total momentum remains zero. But there is no *a priori* reason that the impulse to the capacitor should equal the momentum originally stored in the fields, or that it should be the same when we turn off the electric field as when we turn off the magnetic field.

The cleanest example of hidden momentum is the following: Imagine a rectangular loop of wire carrying a steady current.[14] Picture the current as a stream of noninteracting positive charges that move freely within the wire. When a uniform electric field $\mathbf{E}$ is applied (see Fig. 9), the charges will accelerate up the left segment and decelerate down the right one. Notice that there are fewer charges in the upper segment but they are moving faster. *Question:* What is the total momentum of all the charges in the loop?

The momenta of the left and right segments cancel, so we need only consider the top and bottom segments. Say there are N_t charges in the top segment, going to the right at speed v_t, and N_b charges in the lower segment, going at speed v_b to the left. The current ($I = \lambda v$) is the same in all four segments (otherwise charge would be piling up somewhere, and it would not be a steady current). Thus

Table I. Capacitor in a magnetic field—naive solution.

Momentum initially stored in fields	$-BQd\hat{\mathbf{x}}$
Momentum delivered to capacitor as it discharges	$-BQd\hat{\mathbf{x}}$
Momentum delivered to capacitor as $\mathbf{B}$ decreases	$-BQd\hat{\mathbf{x}}$

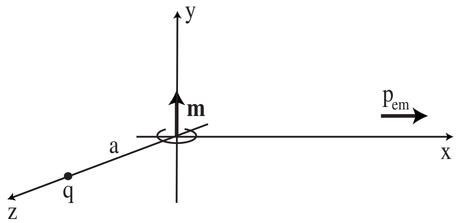

Fig. 3. There is electromagnetic momentum in the fields of a point charge near a magnetic dipole even though both are stationary.

$$I = \frac{qN_t}{l} v_t = \frac{qN_b}{l} v_b, \tag{12a}$$

so

$$N_t v_t = N_b v_b = \frac{Il}{q}, \tag{12b}$$

where q is the charge of each particle and l is the length of the rectangle. *Nonrelativistically,* the momentum of a single particle is $p = mv$, where m is its mass, so the total momentum (to the right) is

$$p_{\text{class}} = mN_t v_t - mN_b v_b = m\frac{Il}{q} - m\frac{Il}{q} = 0, \tag{13}$$

as we would expect (after all, the loop as a whole does not move). But *relativistically* the momentum of a particle is $p = \gamma mv$, and we get

$$p_{\text{rel}} = \gamma_t m N_t v_t - \gamma_b m N_b v_b = \frac{mIl}{q}(\gamma_t - \gamma_b), \tag{14}$$

which is *not* zero because the particles in the upper segment move faster.

As a particle goes up the left side, it gains energy equal to the work done by the electric force,

$$\gamma_t mc^2 - \gamma_b mc^2 = qEw, \tag{15a}$$

so

$$p_{\text{rel}} = \frac{IlEw}{c^2}, \tag{15b}$$

and hence

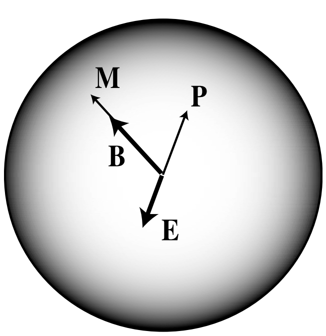

Fig. 4. A sphere with uniform polarization **P** and uniform magnetization **M** carries electromagnetic momentum proportional to $\mathbf{M}\times\mathbf{P}$.

Table II. Capacitor in a magnetic field—corrected.

Momentum initially stored in fields	$-\frac{1}{2}BQd\hat{\mathbf{x}}$
Momentum delivered to capacitor as it discharges	$-BQd\hat{\mathbf{x}}$
Momentum delivered to capacitor as **B** decreases	$-\frac{1}{2}BQd\hat{\mathbf{x}}$

$$\mathbf{p}_{\text{hid}} = \frac{1}{c^2}(\mathbf{m}\times\mathbf{E}), \tag{16}$$

where $\mathbf{m}$ is the magnetic dipole moment of the loop ($|\mathbf{m}| = Ilw$). If we integrate $\wp_{\text{em}} = \epsilon_0(\mathbf{E}\times\mathbf{B})$ for a magnetic dipole in an electric field,[17] we obtain[18]

$$\mathbf{p}_{\text{em}} = -\frac{1}{c^2}(\mathbf{m}\times\mathbf{E}). \tag{17}$$

The hidden momentum exactly cancels the field momentum, as the center of energy theorem requires.

It is easy to generalize this result. In terms of the electric potential V, $(\gamma_t - \gamma_b)mc^2 = -q(V_t - V_b)$, and hence[19]

$$\mathbf{p}_{\text{hid}} = -\frac{mIl}{q}\frac{q}{mc^2}(V_t - V_b)\hat{\mathbf{x}} = -\frac{I}{c^2}(V_t - V_b)l\hat{\mathbf{x}} = -\frac{I}{c^2}\oint V d\boldsymbol{\ell}, \tag{18}$$

where the integral is around the loop in the direction of the current (the ends cancel). For surface or volume currents, we have

$$\mathbf{p}_{\text{hid}} = -\frac{1}{c^2}\int V\mathbf{K}da \tag{19a}$$

and

$$\mathbf{p}_{\text{hid}} = -\frac{1}{c^2}\int V\mathbf{J}d\tau. \tag{19b}$$

This model of electric current is artificial, and one might prefer to treat it as an incompressible fluid.[20] In that case, assuming that the wire has a constant cross section, the speed and spacing of the charges are the same all the way around the loop, but those in the top segment are under higher *pressure*. Now, a moving fluid under high pressure carries greater momentum than the same fluid under low pressure. The quickest way to see this is by examining the stress-energy tensor for a simple fluid,[21]

$$T^{\mu\nu} = \rho_0 v^\mu v^\nu + P\left(\frac{v^\mu v^\nu}{c^2} - g^{\mu\nu}\right), \tag{20}$$

where ρ_0 is the mass density in the rest frame of the fluid, P is the (scalar) pressure, and $v^\mu = \gamma(c, \mathbf{v})$ is the local

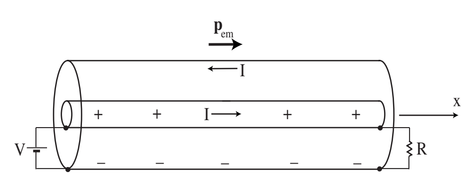

Fig. 5. A coaxial cable, with current flowing from a battery at one end through a resistor at the other and returning, carries electromagnetic momentum in the direction of the high-voltage current.

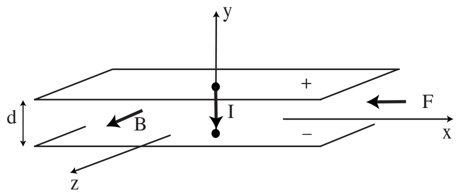

Fig. 6. A fine wire between the plates allows current I to flow, discharging the capacitor. The magnetic force on the current delivers an impulse to the capacitor.

4-velocity. The $0i$ components of $T^{\mu\nu}$ give the momentum density,

$$T^{0i} = \gamma^2\left(\rho_0 c + \frac{P}{c}\right)v^i = \wp^i c. \tag{21}$$

Thus, the momentum density of the fluid is

$$\wp_{\text{fluid}} = \gamma^2\left(\rho_0 + \frac{P}{c^2}\right)\mathbf{v}. \tag{22}$$

The first term represents the ordinary flow of mass; the second is the "extra" momentum associated with pressure.[22] The latter accounts for the hidden momentum,

$$p_{\text{hid}} = (\wp_t - \wp_b)lA = \frac{\gamma^2}{c^2}(P_t - P_b)vlA, \tag{23}$$

where A is the cross-sectional area of the wire.

The difference in pressure between the top and bottom segments is due to the (electric) force on the charges in (say) the left segment of the loop. In relativity, the force per unit volume acting on a fluid is given by[23]

$$f^\mu = \frac{\partial}{\partial x^\nu}T^{\mu\nu}. \tag{24}$$

If x^1 is the vertical direction, then from Eq. (20)

$$f^1 = \frac{dP}{dx^1}\left(\frac{\gamma^2 v^2}{c^2} + 1\right) \tag{25a}$$

or

$$\mathbf{f} = \gamma^2 \nabla P. \tag{25b}$$

Equation (25b) is the (relativistic) relation between the force density and the pressure gradient. In our case the force per unit volume is $\rho\mathbf{E}$, so $\gamma^2\nabla P = \rho\mathbf{E}$, or integrating over the volume of the left segment,

$$\gamma^2(P_t - P_b)A = \rho E A w. \tag{26}$$

Putting Eq. (26) into Eq. (23) and using $I = \rho A v$, we get

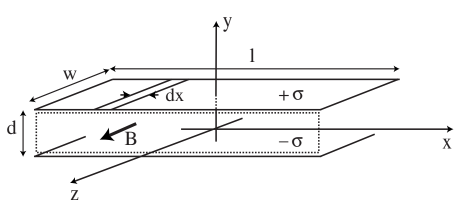

Fig. 7. If the magnetic field is gradually reduced, an electric field is induced, in the counterclockwise direction, imparting an impulse to the capacitor.

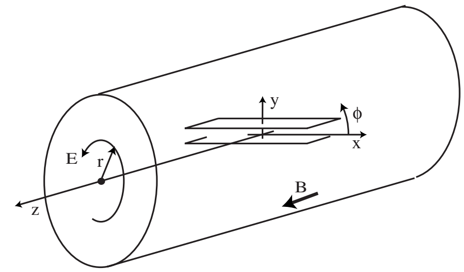

Fig. 8. In this model the magnetic field is produced by a long solenoid, and the induced electric field can be calculated explicitly [Eq. (9)].

$$p_{\text{hid}} = \frac{vl}{c^2}\rho E A w = \frac{IlEw}{c^2}, \tag{27}$$

the same as before [Eq. (15b)].

IV. ELECTRIC DIPOLE AT THE CENTER OF A SPINNING, UNIFORMLY CHARGED SPHERICAL SHELL

What about a capacitor in the magnetic field of a solenoid? In this case the hidden momentum is located in the solenoid (that is where the moving charges are), and the electric field responsible for the variation in γ must be the exterior (fringing) field of the capacitor.[24] The fringing field is notoriously difficult to calculate, and the results are independent of the geometry, so we begin with a simpler model, replacing the capacitor by an electric dipole with dipole moment $\mathbf{p} = p\hat{\mathbf{y}}$,[25] and the solenoid by a spherical shell of radius R which carries a uniform surface charge σ and spins at a constant angular velocity $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$ (see Fig. 10).[26] This configuration produces a uniform magnetic field,

$$\mathbf{B} = \tfrac{2}{3}\mu_0\sigma R\boldsymbol{\omega} \tag{28}$$

for points inside the sphere, and a dipole field

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}], \quad \text{where} \quad \mathbf{m} = \frac{4\pi}{3}\sigma R^4\boldsymbol{\omega} \tag{29}$$

for points outside. The field of the electric dipole is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] - \frac{1}{3\epsilon_0}\mathbf{p}\delta^3(\mathbf{r}). \tag{30}$$

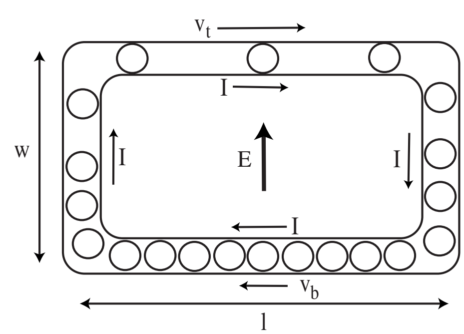

Fig. 9. The simplest model for hidden momentum: a steady current I consisting of noninteracting charges constrained to move around a rectangular tube. In the presence of an external electric field $\mathbf{E}$, the charges accelerate up the left segment and decelerate down the right segment.

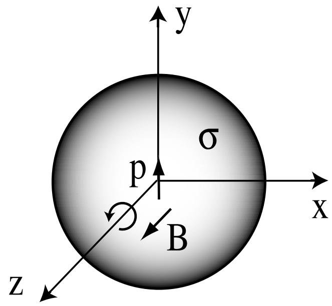

Fig. 10. A uniformly charged spherical shell spins at constant angular velocity, producing a uniform magnetic field at interior points; an electric dipole is located at the center.

To calculate the electromagnetic momentum, we integrate $\epsilon_0(\mathbf{E}\times\mathbf{B})$ over the interior and exterior of the sphere,[27]

$$\mathbf{p}_{\rm in} = \epsilon_0\left\{\int\left[\frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}] - \frac{1}{3\epsilon_0}\mathbf{p}\delta^3(\mathbf{r})\right]d\tau\right\}\times\mathbf{B} \tag{31a}$$

$$= -\frac{1}{3}(\mathbf{p}\times\mathbf{B}). \tag{31b}$$

This part comes exclusively from the delta function because the angular integral of the first term (using spherical coordinates and $\mathbf{p}=p\hat{\mathbf{y}}$),

$$\int[3(p\sin\theta\sin\phi)(\sin\theta\cos\phi\hat{\mathbf{x}}+\sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}) - p\hat{\mathbf{y}}]\sin\theta d\theta d\phi, \tag{32}$$

is zero. The contribution from outside the sphere is

$$\mathbf{p}_{\rm out} = \epsilon_0\frac{1}{4\pi\epsilon_0}\frac{\mu_0}{4\pi}\int\frac{1}{r^6}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}] \times[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}]r^2\sin\theta dr d\theta d\phi \tag{33a}$$

$$= -\frac{\mu_0 mp}{12\pi R^3}\hat{\mathbf{x}} = -\frac{1}{6}(\mathbf{p}\times\mathbf{B}), \tag{33b}$$

where $\mathbf{B}$ is the magnetic field inside the sphere. So

$$\mathbf{p}_{\rm em} = \mathbf{p}_{\rm in}+\mathbf{p}_{\rm out} = -\tfrac{1}{2}(\mathbf{p}\times\mathbf{B}). \tag{34}$$

The hidden momentum in the spinning sphere is [see Eq. (19a)]

$$\mathbf{p}_{\rm hid} = -\frac{1}{c^2}\int V\mathbf{K}da = -\frac{1}{c^2}\int\left(\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}\right)[\sigma(\boldsymbol{\omega}\times\mathbf{r})]da \tag{35a}$$

$$= -\epsilon_0\mu_0\frac{p}{4\pi\epsilon_0}\sigma\omega R\int\sin\theta\sin\phi[\sin\theta\cos\phi\hat{\mathbf{y}} - \sin\theta\sin\phi\hat{\mathbf{x}}]\sin\theta d\theta d\phi \tag{35b}$$

$$= \frac{\mu_0}{4\pi}p\sigma\omega R\left(\frac{4}{3}\pi\hat{\mathbf{x}}\right) = \frac{1}{2}(\mathbf{p}\times\mathbf{B}). \tag{35c}$$

So far, so good: the momentum, $\mathbf{p}_{\rm em}+\mathbf{p}_{\rm hid}$, is zero.

Now let's connect a wire between the ends of the electric dipole, allowing it to discharge. The impulse delivered to the dipole is

$$\mathcal{I}_{\rm dip} = \int\mathbf{F}dt = \int I(\mathbf{d}\times\mathbf{B})dt = \left(\int_q^0 dq\right)\mathbf{d}\times\mathbf{B} = -(\mathbf{p}\times\mathbf{B}), \tag{36}$$

where $\mathbf{d}$ is the separation of the charges, and $\mathbf{p}=q\mathbf{d}$. At the same time, the changing electric field of the dipole induces a magnetic field, which exerts a force on the spinning sphere. To calculate the induced magnetic field, we orient the polar axis along $\mathbf{p}$ (see Fig. 11) and apply the Ampere–Maxwell law ($\oint\mathbf{B}\cdot d\ell=\mu_0 I_{\rm enc}+\mu_0\epsilon_0 d\Phi_E/dt$) to the dotted "Amperian loop" using the spherical cap to determine Φ_E,

$$\Phi_E = \int\mathbf{E}\cdot d\mathbf{a} = \frac{1}{4\pi\epsilon_0}\int\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}]\cdot r^2\sin\theta d\theta d\phi\hat{\mathbf{r}} \tag{37a}$$

$$= \frac{1}{4\pi\epsilon_0 r}\int(2p\cos\theta)\sin\theta d\theta d\phi = \frac{p\sin^2\theta}{2\epsilon_0 r}. \tag{37b}$$

$\mathbf{B}$ points in the $\hat{\phi}$ direction,

$$B(2\pi r\sin\theta) = \mu_0\epsilon_0\frac{d}{dt}\left(\frac{p\sin^2\theta}{2\epsilon_0 r}\right), \tag{38}$$

so

$$B = \frac{\mu_0}{4\pi}\frac{\dot{p}\sin\theta}{r^2} = \frac{\mu_0}{4\pi}\frac{|\dot{\mathbf{p}}\times\hat{\mathbf{r}}|}{r^2}, \tag{39}$$

where $\dot{\mathbf{p}}=d\mathbf{p}/dt$. The force on the spinning sphere (reverting $\mathbf{p}=p\hat{\mathbf{y}}$ and $\boldsymbol{\omega}=\omega\hat{\mathbf{z}}$) is

$$\mathbf{F} = \int(\mathbf{K}\times\mathbf{B})da = \frac{\mu_0\sigma}{4\pi R^2}\int[(\boldsymbol{\omega}\times\mathbf{r})\times(\dot{\mathbf{p}}\times\hat{\mathbf{r}})]da = -\frac{\mu_0\sigma R\omega\dot{p}}{3}\hat{\mathbf{x}}, \tag{40}$$

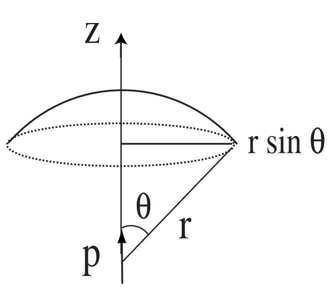

Fig. 11. Geometry for calculating the magnetic field of a discharging electric dipole.

and the impulse delivered to the shell is

Table III. Electric dipole inside spinning sphere—naive solution.

Initial momentum	$\mathbf{p}_{em}=-\frac{1}{2}(\mathbf{p}\times\mathbf{B})$	$\mathbf{p}_{hid}=\frac{1}{2}(\mathbf{p}\times\mathbf{B})$	$\mathbf{p}_{tot}=0$
Dipole discharges	$\mathcal{I}_{dip}=-(\mathbf{p}\times\mathbf{B})$	$\mathcal{I}_{sphere}=\frac{1}{2}(\mathbf{p}\times\mathbf{B})$	$\mathbf{p}_{tot}=-\frac{1}{2}(\mathbf{p}\times\mathbf{B})$
Sphere slows	$\mathcal{I}_{dip}=-\frac{1}{2}(\mathbf{p}\times\mathbf{B})$	$\mathcal{I}_{sphere}=0$	$\mathbf{p}_{tot}=-\frac{1}{2}(\mathbf{p}\times\mathbf{B})$

$$\mathcal{I}_{sphere}=\int \mathbf{F}dt=\frac{1}{2}(\mathbf{p}\times\mathbf{B}). \tag{41}$$

Curiously, this impulse is only half as great as the impulse to the dipole [see Eq. (36)].

Alternatively, we could turn off the magnetic field by bringing the spinning sphere to rest. During this process the changing magnetic field induces an electric field,

$$\mathbf{E}=-\tfrac{1}{2}\dot{B}r\sin\theta\hat{\phi}, \tag{42}$$

which exerts a force on the two ends of the dipole,

$$\mathbf{F}=2q\left(\frac{\dot{B}}{2}\frac{d}{2}\right)\hat{\mathbf{x}}=\frac{1}{2}p\dot{B}\hat{\mathbf{x}}. \tag{43}$$

The net impulse is

$$\mathcal{I}_{dip}=-\tfrac{1}{2}(\mathbf{p}\times\mathbf{B}) \tag{44}$$

(which is the same as the momentum originally stored in the fields). But this time there is *no* electromagnetic impulse to the sphere (see Table III). It is *still* inconsistent! The impulses to the dipole and the spinning sphere do *not* balance, and it seems that the total momentum of the system, after turning off the fields, is not zero.

But wait: What happened to the *hidden* momentum when we turned off the fields? This was purely mechanical momentum associated with the motion of the charges that constitute the current; as the fields are reduced, the excess momentum of the charges is delivered to the structure that keeps them on track (in this case the spherical shell).[28] Thus the hidden momentum should be added to the impulse delivered to the sphere as the fields are turned off, and *finally* everything works out (see Table IV).

V. ELECTRIC DIPOLE IN THE FIELD OF A LONG SOLENOID

The calculations are no more difficult for the solenoid model (see Fig. 12).[29] In this case, the magnetic field is

$$\mathbf{B}(s)=\begin{cases}B\hat{\mathbf{z}} & (s<R)\\ 0 & (s>R),\end{cases} \tag{45}$$

where R is the radius of the solenoid, s is the distance from the axis, and $B=\mu_0 K=\mu_0 nI$ (K is the surface current density, n is the number of turns per unit length, and I is the current). The delta function term in the dipole field in Eq. (30) must be handled with care: this represents the field inside a sphere of radius ϵ (in the limit $\epsilon\to 0$); the "ordinary" dipole field prevails outside this sphere. It is safest to do the calculation in spherical coordinates even though this makes the upper limit on the r integral awkward: $r_{max}=R/\sin\theta$,

Table IV. Electric dipole inside spinning sphere—corrected.

Initial momentum	$\mathbf{p}_{em}=-\frac{1}{2}(\mathbf{p}\times\mathbf{B})$	$\mathbf{p}_{hid}=\frac{1}{2}(\mathbf{p}\times\mathbf{B})$	$\mathbf{p}_{tot}=0$
Dipole discharges	$\mathcal{I}_{dip}=-(\mathbf{p}\times\mathbf{B})$	$\mathcal{I}_{sphere}+\mathbf{p}_{hid}=(\mathbf{p}\times\mathbf{B})$	$\mathbf{p}_{tot}=0$
Sphere slows	$\mathcal{I}_{dip}=-\frac{1}{2}(\mathbf{p}\times\mathbf{B})$	$\mathbf{p}_{hid}=\frac{1}{2}(\mathbf{p}\times\mathbf{B})$	$\mathbf{p}_{tot}=0$

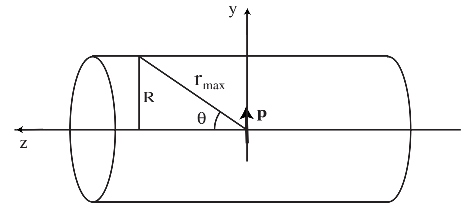

Fig. 12. A long solenoid produces a uniform magnetic field at interior points; an electric dipole is located on the axis.

$$\mathbf{p}_{em}=\epsilon_0\left\{\int\left[\frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}]-\frac{1}{3\epsilon_0}\mathbf{p}\delta^3(\mathbf{r})\right]d\tau\right\}\times\mathbf{B} \tag{46a}$$

$$=\frac{B}{4\pi}\int\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})(\hat{\mathbf{r}}\times\hat{\mathbf{z}})-(\mathbf{p}\times\hat{\mathbf{z}})]d\tau-\frac{1}{3}(\mathbf{p}\times\mathbf{B}). \tag{46b}$$

Now,

$$\mathbf{p}\cdot\hat{\mathbf{r}}=p\sin\theta\sin\phi,$$
$$\hat{\mathbf{r}}\times\hat{\mathbf{z}}=\sin\theta\sin\phi\hat{\mathbf{x}}-\sin\theta\cos\phi\hat{\mathbf{y}},\quad \mathbf{p}\times\hat{\mathbf{z}}=p\hat{\mathbf{x}}, \tag{47}$$

so the integral is

$$p\int\frac{1}{r^3}[3\sin\theta\sin\phi(\sin\theta\sin\phi\hat{\mathbf{x}}-\sin\theta\cos\phi\hat{\mathbf{y}})-\hat{\mathbf{x}}]r^2\sin\theta drd\theta d\phi$$
$$=\pi p\hat{\mathbf{x}}\int_0^{\pi}\left\{\int_{\epsilon}^{R/\sin\theta}\frac{1}{r}dr\right\}(3\sin^2\theta-2)\sin\theta d\theta \tag{48a}$$

$$=\pi p\hat{\mathbf{x}}\int_0^{\pi}[\ln R-\ln(\sin\theta)-\ln\epsilon]\times(1-3\cos^2\theta)\sin\theta d\theta \tag{48b}$$

$$=-\pi p\hat{\mathbf{x}}\int_0^{\pi}\ln(\sin\theta)(1-3\cos^2\theta)\sin\theta d\theta$$
$$=-\frac{2}{3}\pi p\hat{\mathbf{x}}. \tag{48c}$$

Thus

$$\mathbf{p}_{em}=\frac{B}{4\pi}\left(-\frac{2\pi}{3}p\hat{\mathbf{x}}\right)-\frac{1}{3}(\mathbf{p}\times\mathbf{B})=-\frac{1}{2}(\mathbf{p}\times\mathbf{B}), \tag{49}$$

which is the same as for the spinning sphere [see Eq. (34)].

The hidden momentum in the solenoid is

Table V. Electric dipole in the field of a solenoid.

Initial momentum	$\mathbf{p}_{em}=-\frac{1}{2}(\mathbf{p}\times\mathbf{B})$	$\mathbf{p}_{hid}=\frac{1}{2}(\mathbf{p}\times\mathbf{B})$	$\mathbf{p}_{tot}=0$
Dipole discharges	$\mathcal{I}_{dip}=-(\mathbf{p}\times\mathbf{B})$	$\mathcal{I}_{sol}+\mathbf{p}_{hid}=(\mathbf{p}\times\mathbf{B})$	$\mathbf{p}_{tot}=0$
Current decreases	$\mathcal{I}_{dip}=-\frac{1}{2}(\mathbf{p}\times\mathbf{B})$	$\mathbf{p}_{hid}=\frac{1}{2}(\mathbf{p}\times\mathbf{B})$	$\mathbf{p}_{tot}=0$

$$\mathbf{p}_{hid}=-\frac{1}{c^2}\int V\mathbf{K}da=-\mu_0\epsilon_0\int\left(\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}\right)(K\hat{\phi})Rd\phi dz. \tag{50}$$

In this case, it is simplest to use cylindrical coordinates (s,ϕ,z),

$$\mathbf{p}\cdot\hat{\mathbf{r}}=\frac{pR}{r}\sin\phi,\quad r=\sqrt{R^2+z^2},\quad \hat{\phi}=-\sin\phi\hat{\mathbf{x}}+\cos\phi\hat{\mathbf{y}}, \tag{51}$$

so

$$\mathbf{p}_{hid}=-\frac{\mu_0}{4\pi}pKR^2\int\frac{\sin\phi(-\sin\phi\hat{\mathbf{x}}+\cos\phi\hat{\mathbf{y}})}{(R^2+z^2)^{3/2}}d\phi dz \tag{52a}$$

$$=\frac{1}{2}\mu_0 pKR^2\hat{\mathbf{x}}\int_0^\infty\frac{dz}{(R^2+z^2)^{3/2}}$$

$$=\frac{1}{2}pB\hat{\mathbf{x}}=\frac{1}{2}(\mathbf{p}\times\mathbf{B}), \tag{52b}$$

as in the spherical model [Eq. (35)].

Now we discharge the dipole. The impulse to the dipole itself is the same as in Eq. (36), and the induced magnetic field is again given by Eq. (39). The force on the solenoid is

$$\mathbf{F}=\int(\mathbf{K}\times\mathbf{B})da=\frac{\mu_0}{4\pi}\int\frac{1}{r^2}[\mathbf{K}\times(\dot{\mathbf{p}}\times\hat{\mathbf{r}})]Rd\phi dz. \tag{53}$$

But

$$\mathbf{K}\times(\dot{\mathbf{p}}\times\hat{\mathbf{r}})=\frac{\dot{p}K}{r}(-R\cos^2\phi\hat{\mathbf{x}}-R\sin\phi\cos\phi\hat{\mathbf{y}}$$
$$-z\cos\phi\hat{\mathbf{z}}), \tag{54}$$

so

$$\mathbf{F}=-\frac{1}{2}\mu_0\dot{p}KR^2\hat{\mathbf{x}}\int_0^\infty\frac{1}{(R^2+z^2)^{3/2}}dz=-\frac{1}{2}B\dot{p}\hat{\mathbf{x}}. \tag{55}$$

The impulse delivered to the solenoid is

$$\mathcal{I}_{sol}=\int\mathbf{F}dt=-\frac{1}{2}B\hat{\mathbf{x}}\int\frac{dp}{dt}dt$$
$$=-\frac{1}{2}B\hat{\mathbf{x}}\int_p^0 dp=\frac{1}{2}Bp\hat{\mathbf{x}}=\frac{1}{2}(\mathbf{p}\times\mathbf{B}), \tag{56}$$

again reproducing the result for the spinning sphere in Eq. (41).

If we turn off $\mathbf{B}$ (by reducing the current in the solenoid), the induced electric field is the same as in Eq. (42), and the impulse delivered to the dipole is the same as in Eq. (44); again, there is no impulse to the solenoid. The entire table is essentially unchanged (compare Tables IV and V).

VI. CONCLUSION

We considered a rectangular current loop in a uniform electric field, an electric dipole at the center of a spinning charged spherical shell, and an electric dipole in the field of a long solenoid; in each case the hidden momentum exactly balances the electromagnetic momentum, so the total is zero, consistent with the center of energy theorem. Can we prove that this cancellation *always* works when the fields are static? Yes, for if we use Ampere's law ($\nabla\times\mathbf{B}=\mu_0\mathbf{J}$) to replace the current density in the general expression for hidden momentum [see Eq. (19b)] and integrate by parts, we obtain

$$\mathbf{p}_{hid}=-\frac{1}{c^2}\int V\mathbf{J}d\tau$$
$$=-\frac{\epsilon_0\mu_0}{\mu_0}\int V(\nabla\times\mathbf{B})d\tau \tag{57a}$$

$$=-\epsilon_0\int[\nabla\times(V\mathbf{B})+(\mathbf{B}\times(\nabla V)]d\tau \tag{57b}$$

$$=\epsilon_0\oint V\mathbf{B}\times d\mathbf{a}+\epsilon_0\int(\mathbf{B}\times\mathbf{E})d\tau$$
$$=-\epsilon_0\int(\mathbf{E}\times\mathbf{B})d\tau=-\mathbf{p}_{em}. \tag{57c}$$

(For a localized distribution the boundary term is zero.)

When the electric or magnetic field is turned off, the field momentum disappears, the hidden momentum is absorbed, and some element(s) in the system may receive an impulse. But there is no obvious reason why this impulse should equal the momentum originally stored in the fields—all we can say in general is that the total momentum afterward, like the total momentum before, is zero.[30]

The hero (or is it the villain?) of this story is hidden momentum. What can be said about the nature of hidden momentum in general? It seems to share three general features:

- It is purely mechanical.[31] Although it arises most often in electromagnetic contexts, it has nothing to do with electrodynamics. The force involved in Fig. 9 could just as well be gravity, or little rockets attached to the particles, and Eq. (19b) could be written more generally as

$$\mathbf{p}_{hid}=-\frac{1}{c^2}\int u\mathbf{v}d\tau, \tag{58}$$

where u is the potential energy density (of whatever form) and $\mathbf{v}$ is the local velocity.
- It occurs in systems with internally moving parts (such as current loops).[32]
- It is intrinsically relativistic.

A definitive characterization of the phenomenon remains elusive, and some have suggested that the term should be expanded to include *all* strictly relativistic contributions to momentum [including electromagnetic momentum, the $(\gamma-1)mv$ piece of particle momentum, and the $\gamma^2P\mathbf{v}/c^2$ portion of the momentum density of a fluid under pressure]; others urged that the term be expunged altogether.[33]

[1] The relation between momentum density and energy flux is not peculiar to electrodynamics. See R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures* (Addison-Wesley, Reading, MA, 1964), Vol. 2, Eq. (27.21).

[2] F. S. Johnson, B. L. Cragin, and R. R. Hodges, "Electromagnetic momentum density and the Poynting vector in static fields," Am. J. Phys. **62**, 33–41 (1994).

[3] K. T. McDonald, "Electromagnetic momentum of a capacitor in a uniform magnetic field" <www.hep.princeton.edu/~mcdonald/examples/cap_momentum.pdf>.

[4] As we shall see, this turns out to be incorrect.

[5] W. H. Furry, "Examples of momentum distributions in the electromagnetic field and in matter," Am. J. Phys. **37**, 621–636 (1969).

[6] R. H. Romer, "Question #26. Electromagnetic field momentum," Am. J. Phys. **63**, 777–779 (1995). Romer did not use a polarized/magnetized sphere but a sphere carrying the same surface charge and current distributions as they would produce. This avoids the awkward question of whether we should use $\mathbf{D}\times\mathbf{B}$ in place of Eq. (1) and preempts conceptual problems about hidden momentum in bound currents. If such issues arise, this example is to be interpreted in Romer's way, as a configuration of *free* charges and free currents.

[7] Hidden momentum occurs in moving systems as well. See, for instance, E. Comay, "Exposing 'hidden momentum'," Am. J. Phys. **64**, 1028–1034 (1996), but it is most striking in static configurations, and we shall concentrate on such cases.

[8] In this paper all such processes will be carried out quasistatically to avoid electromagnetic radiation (which would remove momentum from the system).

[9] This, incidentally, is the answer provided in the Solution Manual to D. J. Griffiths, *Introduction to Electrodynamics*, 3rd ed. (Prentice-Hall, Upper Saddle River, NJ, 1999), Problem 8.6. The error was caught by David Babson.

[10] See S. Coleman and J. H. Van Vleck, "Origin of 'hidden momentum forces' on magnets," Phys. Rev. **171**, 1370–1375 (1968) and Refs. 5 and 11.

[11] M. G. Calkin, "Linear momentum of the source of a static electromagnetic field," Am. J. Phys. **39**, 513–516 (1971).

[12] Center-of-energy is the natural relativistic generalization of center-of-mass: $\int \mathbf{r} u\, d\tau / \int u\, d\tau$, where u is the energy density.

[13] It is notoriously dangerous to speak of the momentum (or energy) of a configuration that is not localized in space. In this context a uniform field should always be interpreted to mean *locally* uniform, but going to zero at infinity.

[14] See Refs. 11, 15, and 16.

[15] L. Vaidman, "Torque and force on a magnetic dipole," Am. J. Phys. **58**, 978–983 (1990).

[16] V. Hnizdo, "Hidden momentum of a relativistic fluid carrying current in an external electric field," Am. J. Phys. **65**, 92–94 (1997).

[17] D. J. Griffiths, "Dipoles at rest," Am. J. Phys. **60**, 979–987 (1992).

[18] Because electromagnetic momentum [see Eq. (1)] is linear in **B**, this result—expressed in terms of the total magnetic dipole moment—holds for any collection of dipoles, and hence in particular for the rectangular loop (which could be built up as a tessellation of tiny squares).

[19] See Refs. 11, 15, 16, and 3.

[20] See Ref. 15 for the nonrelativistic argument and Ref. 16 for the relativistic version.

[21] See, for example, R. J. Adler, M. J. Bazin, and M. Schiffer, *Introduction to General Relativity*, 2nd ed. (McGraw-Hill, New York, 1975), Sec. 9.2.

[22] *Why* should a moving fluid under pressure carry extra momentum? This is a surprisingly subtle relativistic effect. For a lovely explanation, see K. Jagannathan, "Momentum due to pressure: A simple model," Am. J. Phys. **77**, 432–433 (2009). The essence of the argument appears also in Ref. 7, Sec. IIB. A variation is suggested by K. Szymanski, "On the momentum of mechanical plane waves," Physica B **403**, 2996–3001 (2008), Sec. V; see also R. Medina, "The inertia of stress," Am. J. Phys. **74**, 1031–1034 (2006).

[23] A more general version of the following argument is found in Ref. 16 and in a slightly different form in V. Hnizdo, "Hidden mechanical momentum and the field momentum in stationary electromagnetic and gravitational systems," Am. J. Phys. **65**, 515–518 (1997).

[24] This realization casts doubt on our original naive expression for the field momentum in Eq. (2), which took the electric field to be uniform inside the capacitor and zero outside, and as we have seen, that equation is in error (Ref. 3).

[25] Because electromagnetic momentum [see Eq. (1)] is linear in **E**, our result—expressed in terms of the total electric dipole moment—will hold for any *collection* of dipoles, and hence in particular for the original capacitor model. This is how McDonald (Ref. 3) fixed the error in Eq. (2).

[26] More precisely, we want an electrically neutral spherical shell that carries a surface current density $\mathbf{K}=\sigma(\boldsymbol{\omega}\times\mathbf{r})$. In the presence of an electric field, the charges constituting this current will speed up and slow down (in the first model of Sec. III), spoiling the simple picture of a rigid spinning sphere. The incompressible fluid model is, in this sense, closer in spirit to the spinning sphere.

[27] More simply, we can use the general expression for the momentum of an electric dipole in a magnetic field (Ref. 17): $\mathbf{p}_{\mathrm{em}}=(\mathbf{p}\cdot\nabla)\mathbf{A}$, where in this case $\mathbf{A}=-\frac{1}{2}(\mathbf{r}\times\mathbf{B})$.

[28] This is the "hidden momentum force" that began the whole story. W. Shockley and R. P. James, " 'Try simplest cases' discovery of 'hidden momentum' forces on 'magnetic currents'," Phys. Rev. Lett. **18**, 876–879 (1967).

[29] One might worry, of course, about assuming that the solenoid is infinitely long, or—if finite—the approximations involved in treating its field as uniform inside and zero outside.

[30] The rotational analog is strikingly different: Angular momentum in static fields is not (typically) compensated by hidden angular momentum (there is no rotational analog to the center-of-energy theorem), and when the fields are removed, the system starts to rotate, with angular momentum equal to that originally stored in the fields. The classic case is the "Feynman disk paradox" (Ref. 1, Sec. 17-4). It is easy to construct configurations with hidden angular momentum, but because there is no analog to the center of energy theorem, hidden angular momentum is not *forced* on us as dramatically as hidden linear momentum.

[31] A possible counterexample is given in Ref. 7, Sec. III, where the role of hidden momentum is played by standing electromagnetic waves.

[32] For example, if the magnetic field is produced by stationary magnetic monopoles, instead of electric currents, then there is no hidden momentum, but in that case the electromagnetic momentum also vanishes (Ref. 17).

[33] Although hidden momentum was first discovered 40 years ago (Ref. 28), it continues to carry a mysterious aura and has been misunderstood and misused by a number of authors (including one of us, D.J.G.). Just last year Jon Thaler (personal communication, August 26, 2007) and Timothy Boyer, "Concerning 'hidden momentum'," Am. J. Phys. **76**, 190–191 (2008) pointed out that the coaxial cable is not an example of hidden momentum—there is momentum in the fields, but the center of energy is not at rest. The battery at one end is losing energy, and the resistor at the other end is gaining energy, and the momentum in the fields is precisely the momentum associated with the motion of the center of energy.